VACUUM SPACETIMES WITH AN ISOMETRY

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In vacuum space-times the exterior derivative of a Killing vector field is a two-form that satisfies Maxwell equations without electromagnetic sources. Using the algebraic structure of this two-form we have set up a new formalism for the study of vacuum space-times with an isometry.

1 Introduction

Symmetries play an important role in physics and in particular in General Relativity, where they have played an important role in the search for exact solutions of Einstein's equations.[1,2] Of particular relevance are Killing symmetries, which are generated by vector fields satisfying Killing equations: $\xi_{a;b}+\xi_{b;a}=0$, where the semicolon denotes covariant differentiation. In a given space-time we can look for isometries just by studying the integrability conditions of the Killing equations, which are:

$$\pounds(\boldsymbol{\xi})\Gamma^a{}_{bc} = \pounds(\boldsymbol{\xi})R^a{}_{bcd} = \pounds(\boldsymbol{\xi})R^a{}_{bcd;e_1} = \pounds(\boldsymbol{\xi})R^a{}_{bcd;e_1e_2} = \cdots = 0\,, \qquad (1)$$

where $\pounds(\boldsymbol{\xi})$ means Lie differentiation along the Killing vector field (KVF) $\boldsymbol{\xi}$, $\Gamma^a{}_{bc}$ are the Christoffel symbols, and $R^a{}_{bcd}$ the components of the Riemann tensor. The isometries of a space-time form a Lie group, which provides a way of classifying space-times in terms of the Lie group that they admit. Another important way of classifying space-times is through the algebraic classification of the Weyl tensor (the so-called Petrov classification [1]), C_{abcd}, which, in vacuum, coincides with the Riemann tensor. The relation between both classifications remains still unclear. In this communication we describe a new approach to the study of Killing symmetries in vacuum space-times, based on the introduction of an algebraic structure associated with the KVF, which allows to establish connections between the existence of an isometry and the Petrov type of the space-time.

2 The algebraic structure associated with an isometry

An interesting property of KVFs in vacuum space-times, firstly noticed by Papapetrou,[3] is that their exterior derivative is a 2-form satisfying Maxwell's equations in the absence of electromagnetic currents. This 2-form, which we will call the *Papapetrou* field associated with $\boldsymbol{\xi}$, is given by

$$F_{ab} = \xi_{b;a} - \xi_{a;b} = 2\xi_{b;a}\,. \tag{2}$$

In the same way as we endow space-times with the algebraic structure of the Weyl tensor, we can consider the algebraic structure of F_{ab} as the algebraic structure associated with the KVF $\boldsymbol{\xi}$. The algebraic classification of a 2-form consists of two differentiated cases: (i) The *regular* case, characterized by

$$\tilde{F}^{ab}\tilde{F}_{ab} \neq 0\,, \quad \text{where} \quad \tilde{F}_{ab} \equiv F_{ab} + i * F_{ab}\,, \quad \text{and} \quad * F_{ab} \equiv \tfrac{1}{2}\eta_{abcd}F^{cd}\,,$$

and where $\tilde{}$ and $*$ denote the self-dual and dual operations respectively. In this case we can pick a Newman-Penrose basis $\{\boldsymbol{k}, \boldsymbol{\ell}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$ so that $\tilde{\boldsymbol{F}}$ takes the following *canonical* form

$$\tilde{F}_{ab} = -(\not\alpha + i\,\not\beta)W_{ab}\,, \quad W_{ab} \equiv 2m_{[a}\bar{m}_{b]} - 2k_{[a}\ell_{b]}\,, \tag{3}$$

where $\not\alpha$ and $\not\beta$ are the real eigenvalues of F_{ab}, and $\boldsymbol{\ell}$ and $\boldsymbol{k}$ are its null eigenvectors. (ii) The *singular* case, characterized by $\tilde{F}^{ab}\tilde{F}_{ab} = 0$. Now, we can choose the Newman-Penrose basis so that $\tilde{\boldsymbol{F}}$ can be cast in the form

$$\tilde{F}_{ab} = \phi V_{ab}\,, \quad V_{ab} \equiv 2k_{[a}m_{b]}\,, \tag{4}$$

where ϕ is a complex scalar and $\boldsymbol{k}$ the only principal direction. There is a covariant way [4] of obtaining the eigenvalues and principal direction(s) from the KVF and quantities constructed from it.

Combining the algebraic structures of the Weyl tensor and of the Papapetrou field of a KVF, we can introduce a new classification of the vacuum space-times having at least one KVF, or more precisely, of the pairs $\{(V_4, \boldsymbol{g}), \boldsymbol{\xi}\}$, which takes into account the fact that there are space-times with more than one KVF. Then, we can classify these pairs according to the following properties: The algebraic type of the Papapetrou field (regular or singular); the Petrov type of the space-time (I, II, III, D, N, or O); the degree of alignment of the principal directions of the Papapetrou field with those of the Weyl tensor. Finally, we can refine this classification by adding differential invariant properties of the principal null directions.

3 A formalism for vacuum space-times with an isometry

Starting from the Papapetrou field and its algebraic structure we are going to introduce a new approach, extension of the Newman-Penrose (NP) formalism,[5] to study vacuum space-times with a non-null KVF. This extension consists of two steps: (i) To add new variables related to the KVF, and their corresponding equations. (ii) To write all the equations (those of the NP formalism plus those for the new variables) in a NP basis in which the Papapetrou field of the KVF takes its canonical form [(3) in regular case and (4) in the singular case]. Then, the alignment of a principal direction of the Papapetrou field with one of the Weyl tensor can be study in a natural way within this formalism. For instance, setting $\Psi_0 = 0$ we impose the principal direction of the Papapetrou field $\boldsymbol{k}$ to be aligned with one principal direction of the space-time.

The new variables will be, first, the components of the KVF in a NP basis:

$$\boldsymbol{\xi} = -\xi_l \boldsymbol{k} - \xi_k \boldsymbol{\ell} + \xi_{\bar{m}} \boldsymbol{m} + \xi_m \bar{\boldsymbol{m}}\,,$$

where ξ_l and ξ_k are real and ξ_m complex ($\bar{\xi}_m = \xi_{\bar{m}}$). The equations for these variables come from the definition of the Papapetrou field in terms of the KVF [Eq. (2)], that is, $\xi_{b;a} = \frac{1}{2} F_{ab}$. When we write these equations in a canonical NP basis [6] [See Eqs. (3,4)] we realize that the quantities ($\not\alpha$, $\not\beta$) or ϕ appear. Then, in order to close the system of equations we will consider them as new variables. It turns out that the equations for these variables are the Maxwell equations for the Papapetrou field, i.e., $F^{ab}{}_{;b} = 0$, $F_{[ab;c]} = 0$, which indeed close the system of equations.[6] In the regular case they are:

$$D(\not\alpha + i\, \not\beta) = 2\rho(\not\alpha + i\, \not\beta)\,, \quad \triangle(\not\alpha + i\, \not\beta) = -2\mu(\not\alpha + i\, \not\beta)\,, \tag{5}$$

$$\delta(\not\alpha + i\, \not\beta) = 2\tau(\not\alpha + i\, \not\beta)\,, \quad \bar{\delta}(\not\alpha + i\, \not\beta) = -2\pi(\not\alpha + i\, \not\beta)\,. \tag{6}$$

Once the formalism has been set up, let us see how it works. To that end, we will focus on the regular case ($\not\alpha + i\, \not\beta \neq 0$), since the singular case has already been completely examined [6] and all the spacetimes and the KVFs determined: They correspond to two particular classes of *pp waves* (Petrov type N solutions) and the Minkowski space-time.

One of the things we should look at is the integrability of the equations (2) for the components of the KVF, (ξ_k, ξ_l, ξ_m). The calculations are long but straightforward. The result is that they are integrable if the components of the Weyl tensor are given by the following expressions

$$\Psi_0 = \frac{\not\alpha + i\, \not\beta}{N}(\kappa\xi_m - \sigma\xi_k)\,, \quad \Psi_1 = \frac{\not\alpha + i\, \not\beta}{N}(\kappa\xi_l - \sigma\bar{\xi}_m)\,, \tag{7}$$

$$\Psi_2 = \frac{\not\alpha + i\ \not\beta}{N}(\rho\xi_l - \tau\bar{\xi}_m)\,, \tag{8}$$

$$\Psi_3 = \frac{\not\alpha + i\ \not\beta}{N}(\mu\bar{\xi}_m - \pi\xi_l)\,, \quad \Psi_4 = \frac{\not\alpha + i\ \not\beta}{N}(\nu\bar{\xi}_m - \lambda\xi_l)\,, \tag{9}$$

and the following relations between spin coefficients and components of the KVF hold

$$\kappa\xi_l - \sigma\bar{\xi}_m = \rho\xi_m - \tau\xi_k\,, \quad \rho\xi_l - \tau\bar{\xi}_m = \mu\xi_k - \pi\xi_m\,, \quad \mu\bar{\xi}_m - \pi\xi_l = \nu\xi_k - \lambda\xi_m\,.$$

As we can see, the procedure used in this formalism determines completely the components of the Weyl tensor in terms of spin coefficients and quantities constructed from the KVF, and the dependence on these quantities is algebraic. This contrasts with the usual approach to the integrability conditions for the equations of a KVF which, as we have mention above, involve derivatives of the curvature [see Eq. (1)]. The main features of our procedure that are responsible of this result are: The addition of new variables corresponding to components of the Papapetrou field, the choice of a NP basis in which the Papapetrou field takes its canonical form, and finally, we are using the NP formalism which takes advantage of the fact that the spacetime is a four-dimensional manifold. It can be seen that in higher dimensions the expressions we get for the components of the Weyl tensor do not determine it completely. A direct consequence of the expressions (7-9) for the Weyl tensor is that we do not need to solve the second Bianchi identities, which are the equations for the complex scalars Ψ_A. Instead, we only have to substitute the expressions we have just obtained in the second Bianchi identities to obtain a set of consistency relations.

Taking into account these results, the next step in the application of our formalism will be to study the integrability conditions for the Maxwell equations (5,6), or equivalently, the integrability conditions for the complex quantity $\not\alpha + i\ \not\beta$. Since we know explicitly all the directional derivatives of $\not\alpha + i\ \not\beta$, it is straightforward to study their integrability conditions. We have found [6] that they are additional conditions on the spin coefficients and their directional derivatives. Another step forward in this development is to substitute the expressions (7-9) for the complex scalars Ψ_A into the second Bianchi identities. Using the other equations in our formalism we get expressions containing directional derivatives only of the spin coefficients, that is, we get more restrictions on the spin coefficients.[6] The conclusion we extract from this discussion is that all the integrability and consistency conditions can be reduce to first-order differential equations for the spin coefficients. Specifically they are: (i) The NP equations. (ii) The integrability conditions for the Maxwell equations (5,6). (iii) The consistency conditions coming from the substitution of the expressions (7-9) into the second Bianchi identities.

As is clear, the next step in this study would be to look for the compatibility conditions for the whole set of differential equations for the spin coefficients. In general, the calculations involved are large, but it is possible to study particular cases, specially those in which there are alignments of the principal direction(s) of the Papapetrou field with those of the space-time. For instance, we have studied [6] Petrov type III vacuum space-times arriving at the conclusion that the alignment of the multiple principal direction of the space-time with some of the two principal directions of the Papapetrou field is forbidden. In contrast to this situation, there are other vacuum space-times in which we can find alignments. An interesting example is the case of the Kerr metric in which the two multiple principal directions of the space-time (it is Petrov type D) are aligned with those of the Papapetrou field.[4] A systematic study of alignments in the class of vacuum spacetimes for which $\not\alpha + i \not\beta$ is an analytic function of the Ernst potential has been carried out recently.[7]

To sum up, we have set up a new formalism for the study of vacuum space-times with an isometry. Taking into account the features we have described above, this formalism is suitable for the study of any problem or situation with a Killing symmetry and in which the knowledge of the Weyl complex scalars is required. Some interesting examples are: Search and study of exact solutions, perturbations of black holes preserving a symmetry, the question of the equivalence of metrics, the construction of numerical algorithms, etc.

Acknowledgments

F.F. acknowledges financial support from the D.G.R. of the Generalitat de Catalunya (grant 1998GSR00015), and the Spanish Ministry of Education (contract PB96-0384). C.F.S. is supported by the European Commission (contract HPMF-CT-1999-00149).

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